UNIVERSITÄT KAISERSLAUTERN

ON THE VARIANCE OF THE NUMBER OF PIVOT STEPS REQUIRED BY THE SIMPLEX ALGORITHM

Karl–Heinz Küfer

Preprint Nr. 238

FACHBEREICH MATHEMATIK

ON THE VARIANCE OF THE NUMBER OF PIVOT STEPS REQUIRED BY THE SIMPLEX ALGORITHM

Karl–Heinz Küfer

Preprint Nr. 238

UNIVERSITÄT KAISERSLAUTERN
Fachbereich Mathematik
Erwin–Schrödinger–Straße
6750 Kaiserslautern

Januar 1993

On the Variance of the Number of Pivot Steps Required by the Simplex Algorithm

Karl-Heinz Küfer
Department of Mathematics
University of Kaiserslautern
P.O. Box 3049, D-6750 Kaiserslautern

Introduction: Despite their very good empirical performance most of the simplex algorithm's variants require exponentially many pivot steps in terms of the problem dimensions of the given linear programming problem (LPP) in worst-case situtation. The first to explain the large gap between practical experience and the disappointing worst-case was Borgwardt (1982a,b), who could prove polynomiality on the average for a certain variant of the algorithm—the "Schatteneckenalgorithmus (shadow vertex algorithm)"—using a stochastic problem simulation.
He studied LPP of type

(1) $$\max_{x \in X} v^T x\,,\ X := \{x \in \mathbb{R}^n \,|\, a_i^T x \le 1;\, i = 1,\dots,m\},$$

with $a_i,\ v,\ x \in \mathbb{R}^n,\ m \ge n \ge 2$. The vectors a_i are supposed to be i.i.d.–variables on $\mathbb{R}^n \setminus \{0\}$, whose common distribution is invariant under rotations around the origin. The set X can be considered as a random nonempty polyhedron on $\mathbb{R}^n$. Introducing the notation

(2) $$Y := \{y \in \mathbb{R}^n \,|\, x^T y \le 1;\ x \in X\} = \mathrm{convhull}(0, a_1, \dots, a_m)$$

for the polar polyhedron Y to X we define corresponding to Borgwardt the random variable $s(X)$ by

(3) $$s(X) := \int_{\omega_n}\int_{\omega_n} s_{u,v}(X)\, d^0_{\omega_n}(u)\, d^0_{\omega_n}(v).$$

where

(3.1) $$s_{u,v}(X) := \begin{cases} \text{number of boundary simplices of } Y \text{ intersected by } cone(u,v)\ \text{-1} \\ \quad \text{if } \mathbb{R}^+u \text{ intersects a boundary simplex of } Y \\ \text{number of boundary simplices of } Y \text{ intersected by } cone(u,v) \\ \quad \text{if } \mathbb{R}^+u \text{ doesn't intersect a boundary simplex of } Y \end{cases}$$

$d^0_{\omega_n}(u)$ is the normed differential on the unit sphere ω_n of $\mathbb{R}^n$ in direction of u, i.e. $\int_{\omega_n} d^0_{\omega_n}(u) = 1$.
The above defined number $s_{u,v}(X)$ equals the number of pivot steps, which phase II of the shadow vertex algorithm requires for maximizing the functional $v^T x$ over X when the iteration is started with a vertex of X, whose polar vertex cone intersects $\mathbb{R}^+u$. So, the random variable $s(X)$ is the average number of pivot steps required by phase II of the algorithm to solve an LPP with domain X averaged on the choice of the starting vertex represented by u and the vector v defining the functional to maximize. s is a random variable assigned to the polyhedron, which describes the polyhedron's complexity concerning an LPP.
The most important tool in Borgwardt's above mentioned polynomiality proof is an estimation of the expectation value $E(s)$ of the random variable s. Independent from the underlying rotationally invariant distribution holds, e.g. Borgwardt (1982a,b):

(4) $$\mathrm{E}(s) \le \frac{e\pi}{4}\left(\frac{\pi}{2} + \frac{1}{e}\right) n^3 m^{1/(n-1)}.$$

But knowledge about the expectation value alone doesn't allow to quantify the probability of large deviations of the number of pivot steps from its expectation value. So, many researchers, e.g. Shamir (1987), raised the question for higher moments or even for the distribution of the random variable s.
The aim of the present paper is to answer this question partially for the first time by estimating the quotient $\frac{\mathrm{Var}(s)}{\mathrm{E}^2(s)}$ asymptotically for a subclass of the rotationally symmetric distributions with compact domain.

Main results:

Let

(5) $$F(r) := \begin{cases} \mathrm{P}(\|a\|_2 \leq r) & r \in [0,1) \\ 1 & r \geq 1 \end{cases}$$

be the "radial distribution function (RDF)" associated with a rotationally symmetric distribution on Ω_n, the n-dimensional unit ball, then F belongs to the class of "regularly varying functions at 1", if its tail $\overline{F}(t) := 1 - F(t)$ satisfies:

(6) $$\overline{F}(1-r) \sim r^{\alpha} L(\frac{1}{r}),\ r \to 0+,$$

where $\alpha > 0$ and L is a "slowly varying function at infinity", i.e.:

(6.1) $$L \in \mathcal{L}^1[1,\infty),\ \lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1\ \forall t \in (0,\infty).$$

We call the class $\mathcal{R}$ of rotationally symmetric distributions on Ω_n with regularly varying RDF "distributions with regularly varying behaviour near the boundary of Ω_n". Now we are able to state:

Theorem 1:

i) *For $n \geq 2$ and a distribution of type $\mathcal{R}$:*

(7) $$\frac{\mathrm{Var}(s)}{\mathrm{E}^2(s)} = O\left((1-\tilde{G}(\tfrac{1}{m}))^{\frac{n-1}{2}}\right),\ m \to \infty,$$

where $\tilde{G}$ is the inverse of the function

(7.1) $$G(t) := P(a^{(n)} \geq t),\ t \in [0,1].$$

ii) *For $n \geq 2$ and a distribution of type $\mathcal{R}$, where the function L of the RDF is a constant function, holds:*

(8) $$\frac{\mathrm{Var}(s)}{\mathrm{E}^2(s)} = O(m^{-\frac{n-1}{\alpha+n-1}}),\ m \to \infty.$$

Remark on theorem 1:

By Chebychev's inequality

(9) $$\mathrm{P}(|\frac{s}{\mathrm{E}(s)} - 1| > a) \leq a^{-2}\frac{\mathrm{Var}(s)}{\mathrm{E}^2(s)}$$

it is a consequence of (7) and (8) that even small relative deviations of s from the expectation value $\mathrm{E}(s)$ become rare as m increases.

In order to illustrate theorem 1 ii) we reformulate (8) for two familiar special cases of distributions included, the first having an RDF with $\alpha = 1$ and $L(t) = n$, the latter being a limiting case with $\alpha \longrightarrow 0$ and appropriately chosen functions L.

Corollary:

i) *For the uniform distribution on the ball Ω_n, $n \geq 2$:*

$$\frac{\text{Var}(s)}{\text{E}^2(s)} = O(m^{-\frac{n-1}{n+1}}),\ m \longrightarrow \infty. \tag{10}$$

ii) *For the uniform distribution on the sphere ω_n, $n \geq 2$, holds:*

$$\frac{\text{Var}(s)}{\text{E}^2(s)} = O(\tfrac{1}{m}),\ m \longrightarrow \infty. \tag{11}$$

It is possible to generalize the statement of theorem 1 for all distributions in the unit ball Ω_n in weakened form:

Theorem 2:

For any rotationally symmetric distribution, whose RDF satisfies $F(1) = 1$, holds:

$$\frac{\text{Var}(s)}{\text{E}^2(s)} = o(1),\ m \longrightarrow \infty. \tag{12}$$

Remark on theorem 2:

The convergence rate on the right hand side depends on the special choice of the distribution and it is no possibility to estimate the quotient with a common algebraic bound as theorem 1 ii) suggests.

The research of this article has been part of the author's dissertation, Küfer (1992a), where random variables on stochastic polyhedra of similar but more general type have been investigated. The interested reader is referred to Küfer (1992c) for a survey on results of type (7) under more general assumptions on the random variables.

In the next section we sketch the very difficult and technical proof of theorem 1, within which the main differences to the proof of theorem 2 are indicated. For a more detailed discussion of single steps of proofs the reader is referred to Küfer (1992a,b,c).

Proof of the main theorem—additional results:

The main trick in treating the random variable s is to consider it as a random variable of the bounded polar polyhedron Y, cf. (2), which was already used by Borgwardt in order to derive results on the expectation value. It is an important structural property of s that it can be additively decomposed relative to the boundary simplices of Y via

$$s(Y) = \sum_{I_n \subset \{1,\ldots,m\}} \chi(A_{I_n})\tilde{W}(A_{I_n}), \tag{13}$$

where I_n is an arbitrary subset of indices $1, \ldots, m$ with cardinality n, $\chi(A_{I_n})$ is the characteristic function of convhull$(a_\ell \mid \ell \in I_n)$ being a boundary simplex of the polyhedron Y and

$$\tilde{W}(A_{I_n}) := \frac{1}{4}\sum_{k=1}^{n} W(A_{I_n^k}), \tag{13.1}$$

$$W(A_{I_n^k}) := \frac{|\omega_n \cap \text{convhull}(a_\ell \mid \ell \in I_n^k)|}{|\omega_n|}.$$

In (13.1) $I_n^k := I_n \setminus \{i_k\}$, if i_k is the k-th index in the increasingly ordered set of indices I_n; $|\cdot|$ means Lebesgue-measure of appropriate dimension. Formula (13) was discovered by Borgwardt within his dissertation, Borgwardt (1977). The random variable s is a special case of a class of random variables on stochastic bounded polyhedra, which we call of "additive type". An axiomatic treatment of additive type random variables exploiting structural properties in order to derive integral representations of the first two of their moments, which enable estimations of variances, was established in Küfer (1992a,b).
As first step in our analysis of the quotient $\frac{\text{Var}(s)}{\text{E}^2(s)}$ we look at the first moment of the random variable s and cite:

Lemma 1:

For all rotationally symmetric distributions in $\mathbb{R}_n$, $n \geq 2$, *with density function* f *and* $m \geq n$:

$$\text{E}(s) = \binom{m}{n} \int_0^\infty (1 - G(h))^{m-n} \Lambda_{\tilde{W}}(h) dh \tag{14}$$

with

$$\Lambda_{\tilde{W}}(h) = |\omega_n| \int_{\mathbb{R}^{n-1}}^{(n)} \tilde{W}(b_1, \ldots, b_n) |\det(B)| \prod_{\ell=1}^{n} f((\bar{b}_\ell, h)^T) d\bar{b}_\ell, \tag{14.1}$$

where $B := \left(\begin{pmatrix} \bar{b}_1 & \ldots & \bar{b}_n \\ 1 & \ldots & 1 \end{pmatrix}\right)$ *and* $b_\ell = (\bar{b}_\ell, b_\ell^{(n)})^T$.

Lemma 1 is derived from (13) by exploiting symmetries and performing simultanious rotations of the vectors a_ℓ, which has been proved in details by Borgwardt (1977). The restriction to distributions with density function is done for simplicity of notation, the general case is gained by a simple limiting process.
For an asymptotic analysis for big m in case of distributions with domain Ω_n it is useful to substitute $G(t) = \psi$, because the resulting integral

$$E(s) = \binom{m}{n} \int_0^{0.5} (1 - \psi)^{m-n} u_{\tilde{W}}(\psi) d\psi, \tag{15}$$

where

$$u_{\tilde{W}}(\psi) := \frac{\Lambda_{\tilde{W}}(\tilde{G}(\psi))}{g(\tilde{G}(\psi))}, \tag{15.1}$$

$\tilde{G}$ being the inverse of G and g being the derivative of $-G$, can be considered as a Laplace-type integral. As we know from Watson's lemma, the asymptotic behaviour of E(s) for big m is initimately related to the behaviour of $u_{\tilde{W}}$ near 0. We have:

Lemma 2:

For any distribution in $\mathcal{P}$ *and* $n \geq 2$:

$$u_{\tilde{W}}(\psi) \sim C_{n,\alpha} \psi^{n-1} \left(\frac{\overline{F}(\tilde{G}(\psi))}{\psi}\right)^{1/(n-1)}, \; \psi \to 0+, \tag{16}$$

where $C_{n,\alpha}$ *is an appropriate positive constant.*

Proof:
We prove the lemma for $n \geq 4$, the cases $n = 2$ and $n = 3$ being easier special cases omitted here. Following Borgwardt (1987) for a distribution with density function f and domain Ω_n, $\Lambda_{\tilde{W}}$ satisfies:

$$\Lambda_{\tilde{W}}(h) = n|\omega_n||\omega_{n-1}| \int_0^{\sqrt{1-h^2}} \tilde{\Lambda}_W(r,h) \int_{-\sqrt{1-h^2}}^{\sqrt{1-h^2}} |r-s|\tilde{g}_{h,2}(s)dsdr, \tag{17}$$

where

$$\tilde{\Lambda}_W(r,h) := \int_{\sqrt{1-r^2-h^2}\Omega_{n-2}}^{(n-1)} \det{}^2(C)W(c_1,\ldots,c_{n-1}) \prod_{i=1}^{n-1} f((\bar{\bar{c}}_i,r,h)^T)d\bar{\bar{c}}_i \tag{18}$$

with $C := \begin{pmatrix} \bar{\bar{c}}_1 & \ldots & \bar{\bar{c}}_{n-1} \\ 1 & \ldots & 1 \end{pmatrix}$ and $c_\ell = (\bar{\bar{c}}_\ell, r, h)^T$. Furthermore,

$$\tilde{g}_{h,k}(s) := \frac{|\omega_{n-n-k}|}{|\omega_n|} \int_{\sqrt{h^2+s^2}}^{1} \frac{(r^2-h^2-s^2)^{(n-2-k)/2}}{r^{n-2}} dF(r) \tag{19}$$

for $k \leq n-1$ and where f satisfies $f(||a||_2) = f(a)$ for $a \in \mathbb{R}^n$. Using

$$W(c_1,\ldots,c_{n-1}) \sim \frac{|\det(C)|}{|\omega_{n-1}|(n-2)!}, \tag{20}$$

for $h \to \infty$ and $r \to 0+$, we obtain

$$\tilde{\Lambda}_W(r,h) \sim \frac{1}{|\omega_{n-1}|(n-2)!} \int_{\sqrt{1-r^2-h^2}\Omega_{n-2}}^{(n-1)} |\det{}^3(C)| \prod_{i=1}^{n-1} f((\bar{\bar{c}}_i,r,h)^T)d\bar{\bar{c}}_i. \tag{21}$$

Now, we define for $x \in [0,1]$:

$$\overline{\Lambda}_W(x) := \frac{1}{|\omega_{n-1}|(n-2)!} \int_{\sqrt{1-x^2}\Omega_{n-2}}^{(n-1)} |\det{}^3(C)| \prod_{i=1}^{n-1} f(\sqrt{||\bar{\bar{c}}||^2+x^2})d\bar{\bar{c}}_i. \tag{22}$$

Simultaneous rotations of the truncated vectors $\bar{\bar{c}}_\ell$ lead to:

$$\overline{\Lambda}_W(x) = \frac{|\omega_{n-3}|}{|\omega_{n-1}|(n-2)!} \int_0^{\sqrt{1-x^2}} \overline{\overline{\Lambda}}_W(\sqrt{r^2+x^2}) \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} |r-s|^3\tilde{g}_{x,3}(s)dsdr \tag{23}$$

with $\tilde{g}_{x,3}(s)$ defined as in (19) and

$$\overline{\overline{\Lambda}}_W(y) := \int_{\sqrt{1-y^2}\Omega_{n-3}}^{(n-2)} \det{}^4(\overline{C}) \prod_{i=1}^{n-2} f(\sqrt{||\bar{\bar{\bar{c}}}||^2+y^2})d\bar{\bar{\bar{c}}}_i, \tag{24}$$

where $\overline{C} := \begin{pmatrix} \bar{\bar{\bar{c}}}_1 & \ldots & \bar{\bar{\bar{c}}}_{n-2} \\ 1 & \ldots & 1 \end{pmatrix}$.

Having performed a few transformations we reach formula

$$\overline{\overline{\Lambda}}_W(y) = \int_y^1 {}^{(n-2)} \tilde{D}(\lambda_1,\dots,\lambda_{n-2},y) \prod_{i=1}^{n-2} \frac{|\omega_{n-3}|}{|\omega_n|} \frac{(\lambda_i^2 - y^2)^{(n-5)/2}}{\lambda_i^{n-2}} dF(\lambda_i), \tag{25}$$

where

$$\tilde{D}(\lambda_1,\dots,\lambda_{n-2},y) := \int_{\omega_{n-3}} {}^{(n-2)} \det{}^4 \begin{pmatrix} \sqrt{\lambda_1^2 - y^2 \overline{\overline{c}}_1} & \dots & \sqrt{\lambda_{n-2}^2 - y^2 \overline{\overline{c}}_{n-2}} \\ 1 & \dots & 1 \end{pmatrix} \prod_{i=1}^{n-2} d^0_{\omega_{n-3}}(\overline{\overline{c}}_i). \tag{26}$$

Now, we evaluate the determinant function $\tilde{D}$ along its last row and recieve

$$\tilde{D}(\lambda_1,\dots,\lambda_{n-2},y) = \sum_{j=0}^{[(n-2)!]^4} \alpha_j(n) \prod_{i=1}^{n-2} (\lambda_i^2 - y^2)^{k_{i,j}/2}, \tag{27}$$

the $\alpha_j(n)$ being constants depending on n, which result from integration over the spheres ω_{n-3}. Hereby, the numbers $k_{i,j} \in \{0,\dots,4\}$ fulfill the equation:

$$\sum_{i=1}^{n-2} k_{i,j} = 4(n-3). \tag{28}$$

If we replace $\tilde{D}$ in (25) by (27), after some calculations we obtain with the aid of (28) and the asymptotic formula

$$\int_h^1 (\rho^2 - h^2)^\beta dF(\rho) \sim \overline{F}(h)(1-h^2)\beta(\alpha+\beta+1)B(\alpha+1,\beta+1),\ h \to 1,\ \beta > -1, \tag{29}$$

being a consequence of (6), an asymptotic equivalent for $\overline{\overline{\Lambda}}_W$. It holds:

$$\overline{\overline{\Lambda}}_W(y) \sim C^{(1)}_{n,\alpha} \overline{F}^{n-2}(y)(1-y^2)^{(n^2-3n-2)/2} \tag{30}$$

for $y \to 1$. The constant $C^{(1)}_{n,\alpha}$ only depends on n and α. Inserting (30) into (23) an asymptotic evaluation of the integral yields:

$$\overline{\Lambda}_W(x) \sim C^{(2)}_{n,\alpha} \overline{F}^{n-1}(x)(1-x^2)^{(n^2-2n-2)/2} \tag{31}$$

for $x \to 1$. Finally, we insert (31) into (17) using $\tilde{\Lambda}_W(r,h) = \overline{\Lambda}(\sqrt{r^2+h^2})$, from which similar considerations lead to

$$\Lambda_{\tilde{W}}(h) \sim \tilde{C}_{n,\alpha} \overline{F}^n(h)(1-h^2)^{(n^2-n-3)/2}. \tag{32}$$

The statement of the lemma follows from (15) and by comparison of coefficients of the asymptotic equivalents:

$$\psi \sim \frac{1}{2\sqrt{\pi}} \frac{\Gamma(\alpha+1)\Gamma(\frac{n}{2})}{\Gamma(\alpha+\frac{n+1}{2})} \overline{F}(h)(1-h^2)^{(n-1)/2},\ \psi \to 0+,\ h \to 1, \tag{33}$$

and

$$g_0(\tilde{G}(\psi))) \sim \frac{1}{\sqrt{\pi}} \frac{\Gamma(\alpha+1)\Gamma(\frac{n}{2})}{\Gamma(\alpha+\frac{n+1}{2})} \overline{F}(h))(1-h^2)^{(n-3)/2} \psi \to 0+,\ h \to 1 \tag{34}$$

using the substitute $h = \tilde{G}(\psi)$. The only matter left in order to complete the proof of the lemma is to show the positivity of the coefficients $C_{n,\alpha}$. This is done by use of the asymptotic bounds (37) and (38), which yield the estimation

$$(35) \qquad \frac{n(n-1)}{2} \leq \frac{\Gamma(n)\sqrt{2}}{\Gamma(n-\frac{1}{\alpha+(n-1)/2})} C_{n,\alpha} \leq (n-1)^2\sqrt{n}\left(\frac{2\sqrt{\pi}\Gamma(\alpha+(n+1)/2)}{\Gamma(\alpha+1)\Gamma(\frac{n}{2})}\right)^{1/(n-1)}$$

implying the positivity of $C_{n,\alpha}$.

□

As a consequence of lemmata 1 and 2 it is now an easy matter to prove the following theorem with the aid of a variant of Watson's lemma, cf. Seneta (1976) or Küfer (1992a).

Theorem 3:

For any distribution in $\mathcal{P}$ and $n \geq 2$ there is a positive constant $\tilde{C}_{n,\alpha}$ such that:

$$(36) \qquad \mathrm{E}(s) \sim \tilde{C}_{n,\alpha}(1-\tilde{G}(\tfrac{1}{m}))^{-1/2}.$$

$\tilde{G}$ is the inverse function of G defined in (7.1).

Remarks on theorem 3:

i) By a simple transformation of Borgwardt's theorem 9 in Borgwardt (1987) one achieves for arbitrary distributions in Ω_n that

$$(37) \qquad \mathrm{E}(s) \gtrsim \binom{m}{n}\frac{|\omega_n|}{|\omega_{n-1}|}\frac{n^2(n-1)}{2}\binom{m}{n}\int_0^{0.5}(1-\psi)^{m-n}\psi^{n-1}\frac{d\psi}{\sqrt{1-\tilde{G}^2(\psi)}},\ m \to \infty,$$

and

$$(38) \qquad \mathrm{E}(s) \lesssim \binom{m}{n}\frac{|\omega_n|}{|\omega_{n-1}|}(n-1)^2 n^{3/2}\int_0^1(1-\psi)^{m-n}\psi^{n-1}\left(\frac{\overline{F}(\tilde{G}^2(\psi))}{\psi}\right)^{1/(n-1)}d\psi,\ m \to \infty.$$

Hence, by (37), cf. theorem 13 in Borgwardt (1976), E(s) tends to infinity if m does, a fact upon E(s) we will need later in order to prove (12).

ii) Theorem 3 sharpens results of Borgwardt, who received asymptotic lower and upper asymptotic bounds for the special cases mentioned in the corollary of theorem 1. For these special cases the constant $C_{n,\alpha}$ can be calculated exactly, cf. Küfer (1992a).

Having studied the expectation value of s so far, we now analyse the second moment of the random variable s. Our first lemma on this issue gives an integral representation of E(s) for all rotationally symmetric distributions in $\mathbb{R}^n$, which, similar to the first moment's integral representation, enables an asymptotic estimation in case of distributions with compact domain. The result is a special case of theorem 3.2, Küfer (1992b), setting $Z(A_{I_n}) = \tilde{W}(A_{I_n})$ and $\sigma = 0$.

Lemma 3:

For rotationally symmetric distributions with density function f and $m \geq 2n \geq 4$ holds:

$$(39) \qquad \mathrm{E}(s^2) = \sum_{k=0}^{n} q_k e_{\tilde{W}}(k)$$

with

(39.1) $$q_k = \binom{m}{n}\binom{n}{k}\binom{m-n}{k}$$

and

(39.2) $$e_{\tilde{W}}(k) = \begin{cases} \int_0^\infty \int_0^\infty \int_0^\pi G_{1,1}^{m-n-k}(h_1,h_2,\varphi)\Lambda_{k,\tilde{W}}(h_1,h_2,\varphi)d\varphi dh_1 dh_2 & k \neq 0 \\ \int_0^\infty (1-G)^{m-n}(h)\Lambda_{\tilde{W}^2}(h)dh & k = 0 \end{cases},$$

where

(39.3) $$G_{1,1}(h_1,h_2,\varphi) := \mathrm{P}(a^{(n)} \leq h_1 \wedge \sin\varphi a^{(n-1)} + \cos\varphi a^{(n)} \leq h_2),$$

(39.4) $$\Lambda_{k,\tilde{W}}(h_1,h_2,\varphi) := \frac{|\omega_n||\omega_{n-1}|}{\sin^{2-k}\varphi} \int\limits_{\mathbb{R}^{n-2}}^{(n-k)} \int\limits_{(-\infty,d_2]}^{(k)} \int\limits_{(-\infty,d_1]}^{(k)} \lambda_{k,\tilde{W}}(c_1,\ldots,c_{n+k})\ d\mu_k$$

with

(39.5) $$\lambda_{k,\tilde{W}}(c_1,\ldots,c_{n+k}) := |\det\begin{pmatrix}\overline{C}_0\\ \underline{e}^T\end{pmatrix}|\tilde{W}\begin{pmatrix}\overline{C}_0\\ h_1\underline{e}^T\end{pmatrix}|\det\begin{pmatrix}\overline{C}_k\\ \underline{e}^T\end{pmatrix}|\tilde{W}\begin{pmatrix}\overline{C}_k\\ h_2\underline{e}^T\end{pmatrix}$$

and

(39.6) $$d\mu_k := \prod_{\ell=1}^{k+n} f(c_\ell)\prod_{\ell=1}^{k} dc_\ell^{(n-1)} \prod_{\ell=n+1}^{n+k} dc_\ell^{(n-1)} \prod_{\ell=1}^{n+k} d\overline{c}_\ell.$$

The matrices $\overline{C}_\ell$ take the form

$$\overline{C}_0 = \begin{pmatrix}\overline{c}_1 & \ldots & \overline{c}_k & \overline{\overline{c}}_{k+1} & \ldots & \overline{\overline{c}}_n \\ & & & d_1 & \ldots & d_1\end{pmatrix}$$

and

$$\overline{C}_k = \begin{pmatrix}\overline{\overline{c}}_{k+1} & \ldots & \overline{\overline{c}}_n & \overline{c}_{n+1} & \ldots & \overline{c}_{n+k} \\ d_2 & \ldots & d_2 & & & \end{pmatrix},$$

whereas $\underline{e} = (1,\ldots,1)^T \in \mathbb{R}^n$ and

(39.7) $$d_1 := \frac{h_2 - h_1\cos\varphi}{\sin\varphi},\ d_2 := \frac{h_1 - h_2\cos\varphi}{\sin\varphi}.$$

At first sight lemma 3 looks frightening because of its complexity and $\mathrm{E}(s^2)$ seems hardly computable. Therefore, as a first step in case of distributions with domain Ω_n we estimate the main quantities $e_{\tilde{W}}(k)$ by simpler ones, which allow asymptotic estimations similar to those in the proof of lemma 2. To this end, let

(40) $$r := \sqrt{d_1^2 + h_1^2} = \sqrt{d_2^2 + h_2^2}.$$

In geometric respect, r represents the distance of the origin to the intersection point of the lines $a^{(n)} = h_1$ and $\sin\varphi a^{(n-1)} + \cos\varphi a^{(n)} = h_2$ in the plane spanned by the unit vectors e_{n-1} and e_n. It is useful to split the domain of integration in the representation (39.2) of $e_{\tilde{W}}(k)$ in regions where $r > 1$ and $r \leq 1$ respectively, for if we set

(41)
$$\tilde{e}_{\tilde{W}}(k) := \int_0^1 (1 - G(h_1))^{m-n-k} R_{k,\tilde{W}}(h_1) dh_1.$$

(41.1)
$$R_{k,\tilde{W}}(h_1) := 2\int_0^{\pi} \chi(r \leq 1) \int_{h_1}^{1} \Lambda_{k,\tilde{W}}(h_1, h_2, \varphi) dh_2 d\varphi = 2\int_{h_1}^{1} \int_{\arccos h_1 - \arccos h_2}^{\arccos h_1 + \arccos h_2} \Lambda_{k,\tilde{W}}(h_1, h_2, \varphi) d\varphi dh_2,$$

for $k := 1, \ldots, n$ and

(42)
$$\overline{e}_{\tilde{W}} := \int_0^1 \int_0^1 (1 - G(h_1) - G(h_2))^{m-2n} \Lambda_{\tilde{W}}(h_1) \Lambda_{\tilde{W}}(h_2) dh_1 dh_2$$

we can state:

Lemma 4:

For all distributions in Ω_n, $n \geq 2$:

(43.1)
$$e_{\tilde{W}}(k) \leq \tilde{e}_{\tilde{W}}(k), \; k = 1, \ldots, n-1,$$

(43.2)
$$e_{\tilde{W}}(n) \leq \overline{e}_{\tilde{W}} + \tilde{e}_{\tilde{W}}(n).$$

The quantities $\tilde{e}_{\tilde{W}}(k)$ and $\overline{e}_{\tilde{W}}$ look more friendly and enable again an application of Watson-type results concerning Laplace-integrals, because the difficult quantity $G_{1,1}$ has been eliminated.

Proof:
The function $G_{1,1}$, cf. (39.3), satisfies

(44)
$$G_{1,1}(h_1, h_2, \varphi) = 1 - G(h_1) - G(h_2), \; r > 1,$$

and

(45)
$$G_{1,1}(h_1, h_2, \varphi) \leq \min(1 - G(h_1), 1 - G(h_2)), \; r \leq 1,$$

as is geometrically seen. Thus, by use of the functions' $\Lambda_{k,\tilde{W}}$ symmetry in the first two of their variables, (43.1) is a bound for the integral (39.2), if the domain of integration is restricted to the set of triples (h_1, h_2, φ) for which $r \leq 1$. In case of $k = 1, \ldots, n-1$ triples (h_1, h_2, φ) with $r > 1$ cannot occur. Hence, (43.1) estimates $e_{\tilde{W}}(k)$ for $k = 1, \ldots, n-1$. In case of $k = n$ for the second part of the integral's domain containing triples (h_1, h_2, φ), for which $r > 1$, we replace the function $G_{1,1}$ as indicated in (44). Estimating $\chi(r > 1)$ by 1 and using the identities

(46)
$$\Lambda_{n,\tilde{W}}(h_1, h_2, \varphi) = \frac{|\omega_{n-1}|}{|\omega_n|} \sin^{n-2} \varphi \Lambda_{\tilde{W}}(h_1) \Lambda_{\tilde{W}}(h_2)$$

and

(47)
$$|\omega_{n-1}| \int_0^{\pi} \sin^{n-2} \varphi d\varphi = |\omega_n|,$$

we obtain the bound $\overline{\epsilon}_{\tilde{W}}$ for the integral's second part implying (43.2).

□

Finally lemma 5 establishes asymptotic bounds for the quantities $\overline{\epsilon}_{\tilde{W}}$ and $\tilde{\epsilon}_{\tilde{W}}(k)$, $k = 1, \dots, n$ introduced above completing the proof of theorem 1. It holds:

Lemma 5:

For distributions of class $\mathcal{R}$ and $n \geq 2$:
i)

(48) $$\frac{\overline{e}_W}{\mathrm{E}^2(s)} - 1 = O(\tfrac{1}{m}),\ m \to \infty.$$

ii)

(49) $$\frac{\tilde{e}_W(k)}{\mathrm{E}^2(s)} = O\left((1 - \tilde{G}(\tfrac{1}{m})^{(n-1)/2}\right),\ m \to \infty.$$

Proof:
i) If we substitute $G(h_1) = \psi_1$ and $G(h_2) = \psi_2$ in (42), claim (48) is a direct consequence of the technical result

(50) $$\int_0^{0.5}\int_0^{0.5} K(\psi_1, \psi_2)\tilde{u}_{\tilde{W}}(\psi_1)\tilde{u}_{\tilde{W}}(\psi_2)d\psi_1 d\psi_2 = O(\tfrac{1}{m}),\ m \to \infty,$$

where

(50.1) $$\tilde{u}_{\tilde{W}}(\psi_\ell) := \frac{(1-\psi_\ell)^{m-n}u_{\tilde{W}}(\psi)}{\int_0^{0.5}(1-\psi)^{m-n}u_{\tilde{W}}(\psi)d\psi},\ \ell = 1, 2,$$

and

(50.2) $$K(\psi_1, \psi_2) := \frac{(1-\psi_1-\psi_2)^{m-2n}}{(1-\psi_1)^{m-n}(1-\psi_2)^{m-n}}.$$

The proof of (50) is explicitly done within chapter 3 in Küfer (1992c).
ii) Let us first look at the general case $k = 1, \dots, n$, $n \geq 4$. Like in lemma 2 the easier special cases $n = 2$ and $n = 3$ are omitted. By defintion (13.1) and (39.5):

(51) $$\tilde{W}\begin{pmatrix} \overline{C}_0 \\ h_1\underline{e}^T \end{pmatrix} = \frac{1}{4}\sum_{i=1}^{n} W\begin{pmatrix} \overline{C}_0^{(i)} \\ h_1\underline{\overline{e}}^T \end{pmatrix},$$

where $\overline{C}_0^{(i)}$ is received by deleting the i-th column of $\overline{C_0}$ for $i = 1, \dots, n$. For all $\overline{b}_i \in \sqrt{1-h^2}\Omega_{n-1}$, $h \in (0,1]$, W is bounded from above by

(52) $$W\begin{pmatrix} \overline{b}_1 & \dots & \overline{b}_n \\ h & \dots & h \end{pmatrix} \leq \frac{1}{h^{n-1}|\omega_{n-1}|}|\mathrm{convhull}(\overline{b}_1, \dots, \overline{b}_{n-1})|,$$

as is seen by geometric insight. The vectors $\overline{b}_i$ may lie in the intersection of the ball $\sqrt{1-h^2}\Omega_{n-1}$ and a hyperplane possessing the distance $s \leq \sqrt{1-h^2}$ to the origin. Then

(53) $$|\mathrm{convhull}(\overline{b}_1, \dots, \overline{b}_{n-1})| \leq (1-h^2-s^2)^{(n-2)/2}|\tilde{S}| \leq (1-h^2)^{n-2}|\tilde{S}|,$$

where $\tilde{S}$ is the uniquely determined simplex of maximal volume in the ball Ω_{n-2}. Furthermore, we have:

(54) $$|\det(\overline{b}_1,\ldots,\overline{b}_{n-1}|) = s\,(n-2)!\,|\mathrm{convhull}(\overline{b}_1,\ldots,\overline{b}_{n-1})| \leq (n-2)!(1-h^2)^{(n-1)/2}.$$

Hence, for $h_1 \longrightarrow 1$:

(55) $$\tilde{W}\begin{pmatrix} \overline{C}_0 \\ h_1\underline{e}^T \end{pmatrix} = O((1-h_1^2)^{n-2})$$

and

(56) $$|\det\begin{pmatrix} \overline{C}_0 \\ \underline{e}^T \end{pmatrix}| \leq \sum_{i=1}^{n} |\det(\overline{C}^{(i)}| = O((1-h_1^2)^{(n-1)/2}).$$

Analogously, one derives the same results for $\tilde{W}\begin{pmatrix} \overline{C}_k \\ h_2\underline{e}^T \end{pmatrix}$ and $|\det\begin{pmatrix} \overline{C}_k \\ \underline{e}^T \end{pmatrix}|$, if one replaces h_1 by h_2 on the right hand side of (55) and (56). Thus, by (39.5) for $h_1 \leq h_2 \leq 1$:

(57) $$\lambda_{k,\tilde{W}}(c_1,\ldots,c_{n+k}) = O((1-h_1^2)^{(4n-6)/2}),\ h \rightarrow 1.$$

Having studied the kernel $\lambda_{k,\tilde{W}}$ of $\Lambda_{k,\tilde{W}}$ so far, we use the obvious estimation:

(58) $$\int\limits_{\mathbb{R}^{n-2}}^{(n+k)} \int\limits_{(-\infty,d_1]}^{(k)} \int\limits_{(-\infty,d_2]}^{(k)} d\mu_k \leq \tilde{g}_{0,2}^{n-k}(r)\tilde{g}_{0,1}^{k}(h_1)\tilde{g}_{0,1}^{k}(h_2),$$

$\tilde{g}_{0,k}$ being defined by (19), in order to obtain:

(59) $$\frac{\Lambda_{k,\tilde{W}}(h_1,h_2,\varphi)}{\sin^{k-2}\varphi} = O((1-h_1^2)^{p_k/2}\overline{F}^{n+k}(h_1)),\ h_1 \rightarrow 1,$$

for $k = 1,\ldots,n$ and $h_1 \leq h_2 \leq r \leq 1$ with $p_k := (n+k-1)+(n-3)+(n-k-4)$. (59) is proved by the aid of the asymptotic relation $\tilde{g}_{0,k}(t) = O((1-t^2)^{(n-2-k)/2}\overline{F}(t)),\ t \rightarrow 1$. Now, we are enabled to estimate $R_{k,\tilde{W}}$, cf. (41.1). We obtain:

(60) $$R_{k,\tilde{W}}(h_1) = O((1-h_1^2)^{(p_k+k+1)/2}\overline{F}^{n+k}(h_1)),\ h_1 \rightarrow 1,$$

making use of

(61) $$\int\limits_{h_1}^{1} \int\limits_{\arccos h_1 - \arccos h_2}^{\arccos h_1 + \arccos h_2} \sin^{k-2}\varphi d\varphi dh_2 = O((1-h_1^2)^{k/2}),\ h_1 \rightarrow 1$$

If we subsitute $h_1 = \tilde{G}(\psi)$ and use (33) and (34), we receive

(62) $$R_{k,\tilde{W}}(\tilde{G}(\psi)) = O\left(g(\tilde{G}(\psi))\psi^{n+k-1}(1-\tilde{G}^2(\psi))^{(n-3)/2}\right),$$

which leads to the desired bound

(63) $$\tilde{e}_{\tilde{W}}(k) = O\left((1-\tilde{G}(\tfrac{1}{m}))^{(n-3)/2}\right),\ m \rightarrow \infty,$$

by use of a Watson-type result like in the proof of theorem 3 implying claim (49) for $k = 1,\ldots,n$. In order to conclude the proof of part ii) we have to investigate $e_{\tilde{W}}(0)$. With the aid of the same methods as above the integral represenation (14.1) can be asymptotically estimated by

(64) $$\Lambda_{\tilde{W}^2}(h) = O((1-h^2)^{n^2-3}\overline{F}^{n}(h)),\ h \rightarrow 1.$$

Hence, by substitute $h = \tilde{G}(v)$:

$$(65) \qquad \Lambda_{\tilde{W}^2}(\tilde{G}(v)) = O(g(\tilde{G}(v))v^{n-1}(1-\tilde{G}^2(v))^{(n-3)/2}), \; v \to 0+,$$

which leads to the same bound for $e_{\tilde{W}}(0)$ as is established for $\tilde{e}_{\tilde{W}}(k)$ in (63).

□

Remark on lemma 5:

In case of the more general assumption concerning the underlying class of distributions of theorem 2 the asymptotic bounds in both of (48) and (49) have to be weakened to $o(1)$. In this situation, claim i) is shown in weakened form again by the aid of the bilinear functional defined in (50), which can be estimated by terms tending to zero as m tends to infinity. A well-known result on Laplace-type integrals yields that the decrease in m can be arbitrarily slow. The analysis of $\tilde{e}^k_{\tilde{W}}$ must be fully revised, because bounds (55) and (56) are too rough. By a very careful evaluation of $\tilde{W}$ using determinant-type bounds and exploiting Hadarmard's formula one can show, that $\tilde{e}_{\tilde{W}}(k)$ is bounded. So, by remark i) on theorem 3 theorem 2 is fully proved.

References:

[1] Borgwardt, K. H. (1977) *Untersuchungen zur Asymptotik der mittleren Schrittzahl von Simplexverfahren in der linearen Optimierung.* Dissertation, Universität Kaiserslautern.

[2] Borgwardt, K. H. (1982) *Some Distribution-Independent Results About the Asymptotic Order of the Average Number of Pivot Steps of the Simplex Method.* Math. of O. R. 7: 441-462.

[3] Borgwardt, K. H. (1982) *The Average Number of Pivot Steps Required by The Simplex Method is Polynomial.* Z. f. O. R.: 157-177.

[4] Borgwardt, K. H. (1987) *The Simplex Method—A Probabilistic Analysis.* Springer, Berlin.

[5] Küfer, K. H. (1992a) *Asymptotische Varianzanalysen in der stochastischen Polyedertheorie.* Dissertation, Universität Kaiserslautern.

[6] Küfer, K. H. (1992b) *A Simple Integral Representation for the Second Moments of Additive Random Variables on Stochastic Polyhedra.* Preprint Nr. 223, Universität Kaiserslautern.

[7] Küfer, K. H. (1992c) *On the Variance of Additive Random Variables on Stochastic Polyhedra.*, Preprint Nr. 233, Universität Kaiserslautern.

[8] Seneta, E. (1976) *Regularly Varying Functions.* Dold, Eckmann (ed.) (Lecture notes in Mathematics, vol. 508). Springer, New York, Berlin, Heidelberg.

[9] Shamir, R. (1987) *The Efficiency of the Simplex Method: A Survey* Management Science 33: 241-262.